2019-Oct-29 (lecture)

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Outline:

- Simple harmonic motion
- RLC circuits

Last time:

Variation of parameters: $y''+a_1y'+a_0y=Q(x)$, homog. y_1, y_2

$y_p=u_1y_1+u_2y_2$, where u_1, u_2 are functions and

$$u_1'y_1+u_2'y_2=0$$

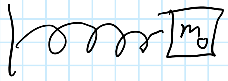
$$u_1'y_1'+u_2'y_2'=Q(x)$$

note, our equation had $a_2=1$, otherwise need $\frac{Q(x)}{a_2}$ here.

Reduction of order: possible to find 1 additional linearly ind. sol if we already know n-1 linearly ind. sol. for order n linear ODE

This time:

- Recall we studied simple harmonic motion earlier



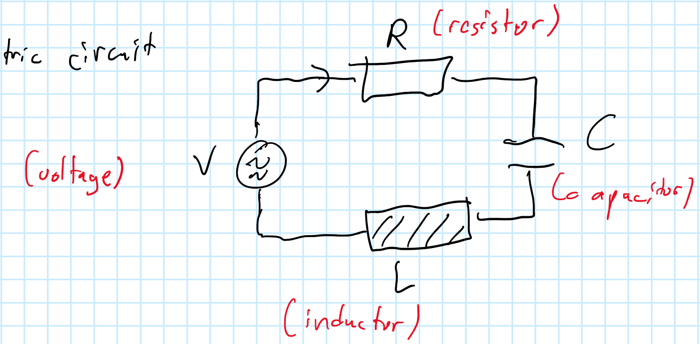
$$m_0\ddot{x}+\underbrace{2m_0r}_{\text{coefficient of resistance}}\dot{x}+m_0w_0^2x=m_0\cdot f(t)\Rightarrow\ddot{x}+2r\dot{x}+w_0^2x=f(t)$$

W_0 ~ natural (undamped) frequency

Char. eq. $m^2+2rm+w_0^2=0$

$$m=-r\pm\sqrt{r^2-w_0^2}$$

- Let's consider a simple RLC electric circuit



Physics: $I(t)$ = current

$V(t)$ = voltage difference

$$\left.\begin{aligned} L\dot{I}_L &= V_L \\ C\dot{V}_C &= I_C \\ V_R &= R I_R \quad \text{(Ohm's Law)} \end{aligned}\right\} \qquad \begin{aligned} L\ddot{I}_L &= \dot{V}_L \\ \frac{I_C}{C} &= \dot{V}_C \\ \dot{V}_R &= R\dot{I}_R \end{aligned}$$

Kirchhoff's 1st Law: $I_L = I_C = I_R$

Kirchhoff's 2nd Law: $V_R + V_L + V_C = V$

$$\dot{V}_R + \dot{V}_L + \dot{V}_C = \dot{V}$$

$$R\dot{I} + L\ddot{I} + \frac{I}{C} = \dot{V}$$

$$\left(\begin{aligned} \text{Note } & L \geq 0 \\ & C \geq 0 \\ & R \geq 0 \end{aligned} \right)$$

2nd order ODE: $L\ddot{I} + R\dot{I} + \frac{1}{C} I = \dot{V}$

Let $\eta = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC} \cdot I = \frac{\dot{V}}{L}$$

$$\ddot{I} + 2\eta\dot{I} + \omega_0^2 I = \frac{\dot{V}}{L}$$

$$\left(\begin{aligned} &\text{harmonic motion} \\ &\ddot{x} + 2r\dot{x} + \omega_0^2 x = f(t) \end{aligned} \right)$$

The RLC system and forced damped harmonic motion have the same behavior.

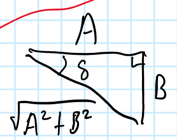
η and r are damping terms, determined by friction, or by the resistor & inductor.

What if $\eta = 0$ $(r = 0)$ i.e. no damping

$$\ddot{x} + \omega_0 x = 0$$

Then $x_c = A\cos(\omega_0 t) + B\sin(\omega_0 t) = \mathrm{Re}\left(Ae^{i\omega_0 t} - iBe^{i\omega_0 t}\right)$

$$= \mathrm{Re}\left[(A - Bi)e^{i\omega_0 t}\right] = \mathrm{Re}\left(\sqrt{A^2+B^2}\, e^{(-\tan^{-1}\frac{B}{A})i} \cdot e^{i\omega_0 t}\right)$$



, $\tan\delta = \frac{-B}{A}$

$$\delta = -\tan^{-1}\frac{B}{A}$$

$\sqrt{A^2+B^2}$

$\delta = -\tan^{-1}\frac{B}{A}$

$$= \sqrt{A^2+B^2} \cdot \mathrm{Re}\left(e^{i\left(\omega_0 t - \tan^{-1}\frac{B}{A}\right)}\right)$$

$$= \sqrt{A^2+B^2} \cdot \cos\left(\omega_0 t - \tan^{-1}\frac{B}{A}\right)$$

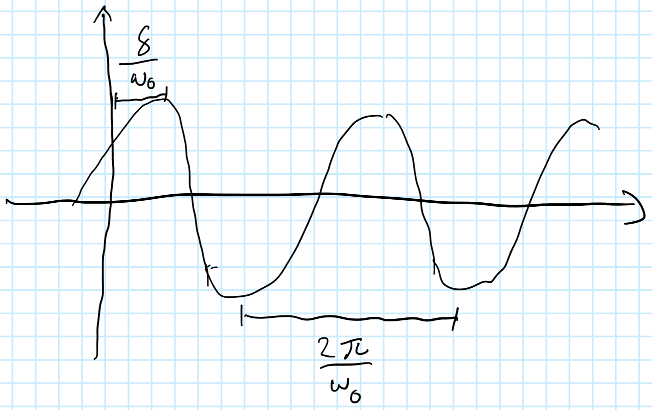
$$= C \cdot \cos(\omega_0 t - \delta),$$

δ is a phase shift

C is amplitude

ω_0 is the resonance (or natural undamped) angular frequency (in radians)

(or $\frac{\omega_0}{2\pi}$ is the resonance frequency in cycles/sec)



never decays

What if $\eta > 0$ $(r > 0)$

$$\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = 0$$

Char. eq. $m^2 + 2\eta m + \omega_0^2 = 0$

$$m = -\eta \pm \sqrt{\eta^2 - \omega_0^2}$$

3 cases:

$\eta > \omega_0$

2 distinct real roots $m_1, m_2 < 0$

$x_c = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$\eta = \omega_0$

1 distinct real root $m = -\eta$

$x_c = (c_1 + c_2 t) e^{-\eta t}$

$\beta = \sqrt{\omega_0^2 - \eta^2}$

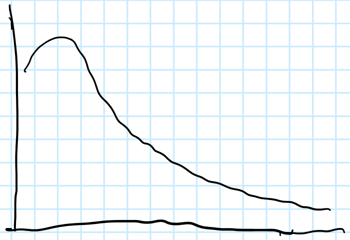
$\eta < \omega_0$

2 conjugate complex roots $m_1, m_2 = -\eta \pm \beta i$

$x_c = c_1 e^{-\eta t}\cos(\beta t) + c_2 e^{-\eta t}\sin(\beta t)$

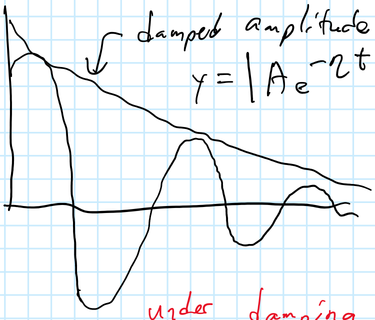
$= A e^{-\eta t}\cos(\beta t - \delta)$

$$= A e^{-\eta t} \cos(\beta t - \delta)$$



Over damping

critical damping



damped amplitude $y = |Ae^{-\eta t}|$

under damping

damped period is $\frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega_0^2 - \eta}}$

The homogeneous soln always decays to 0 exponentially fast if there is damping.

Known as **transient motion** because always decays

Forcing function $\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = f(t)$

- Suppose constant force $f(t) = 1$

  Method of undetermined coeff

$$\left.\begin{aligned} x_p &= c_1 \\ \dot{x}_p &= 0 \\ \ddot{x}_p &= 0 \end{aligned}\right\} \left.\begin{aligned} \omega_0^2 c_1 &= 1 \\ c_1 &= \frac{1}{\omega_0^2} \end{aligned}\right\} x_p = \frac{1}{\omega_0^2}$$

- Suppose periodic forcing term $f(t) = \cos(\omega t)$

  Solve by using complex numbers $\cos(\omega t) = \mathrm{Re}(e^{i\omega t})$

$$x_p = ke^{i\omega t} \rightarrow -\omega^2 k e^{i\omega t} + 2i\omega\eta k e^{i\omega t} + \omega_0^2 k e^{i\omega t} = e^{i\omega t}$$

$$-\omega^2 k + 2i\omega\eta k + \omega_0^2 k = 1$$

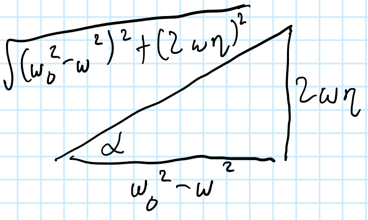
$$k = \frac{1}{(\omega_0^2 - \omega^2) + i(2\omega\eta)}$$

$$x_p = \frac{1}{(\omega_0^2 - \omega^2) + i(2\omega\eta)} e^{i\omega t}$$

$$x = \overset{\text{transient motion}}{x_c} + \overset{\text{steady state motion}}{x_p} \approx x_p \quad \text{as time} \to \infty \quad \text{because } x_c \to 0 \text{ exponentially fast.}$$

$$Re\,(x_p) = Re\left(\frac{(\omega_0^2-\omega^2) - i\,(2\omega\eta)}{(\omega_0^2-\omega^2)^2 + (2\omega\eta)^2} \cdot \left[\cos(\omega t) + i\sin(\omega t)\right]\right)$$

Let



$$\sin\alpha = \frac{2\omega\eta}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}}$$

$$\cos\alpha = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}}$$

$$= Re\left(\frac{\cos\alpha - i\sin\alpha}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}} \cdot \left[\cos(\omega t) + i\sin(\omega t)\right]\right)$$

$$= \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}} \cdot \left(\cos\alpha\cos(\omega t) + \sin\alpha\sin(\omega t)\right)$$

$$x_p = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}} \cdot \cos(\omega t - \alpha) \quad \leftarrow \text{trig identity}$$

So the particular solution is periodic and has the same frequency as the forcing function and amplitude

$$A = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}}$$

If we have $\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = F\cos(\omega t)$, by linearity

$$x_p = \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+(2\omega\eta)^2}} \cdot \cos(\omega t - \alpha)$$

What if the forcing function has the same frequency ω as the $\overset{\text{undamped}}{\text{natural}}$ frequency ω_0?

If no damping: $\ddot{x} + \omega_0^2 x = Fe^{i\omega_0 t}$ (or $\cos\omega_0 t$)

$-i\omega_0 t$ $\quad i\omega_0 t$

If no damping: $\ddot{x} + \omega_0^2 x = Fe^{i\omega_0 t}$ (or $\cos\omega_0 t$)

$m = \pm i\omega_0$, homog. soln. $e^{-i\omega_0 t}$, $e^{i\omega_0 t}$

Then $x_p = \underline{k_1 e^{i\omega_0 t}} + \underline{k_2} t e^{i\omega_0 t}$

$\dot{x}_p = i\omega_0 k_1 e^{i\omega_0 t} + i\omega_0 k_2 t e^{i\omega_0 t} + k_2 e^{i\omega_0 t}$

$\ddot{x}_p = \underline{-\omega_0^2 k_1 e^{i\omega_0 t}} + 2i\omega_0 k_2 e^{i\omega_0 t} \underline{- \omega_0^2 k_2 t e^{i\omega_0 t}}$

red terms cancel out when substituted into $\ddot{x} + \omega_0^2 x = Fe^{i\omega_0 t}$

$\Rightarrow \quad 2i\omega_0 k_2 e^{i\omega_0 t} = Fe^{i\omega_0 t}$

$2i\omega_0 k_2 = F \Rightarrow k_2 = \frac{F}{2i\omega_0} = \frac{-iF}{2\omega_0}$

$x_p = \frac{-iF}{2\omega_0} t e^{i\omega_0 t} = \frac{-iF}{2\omega_0} \cdot t \cdot (\cos(\omega_0 t) + i\sin(\omega_0 t))$

$Re(x_p) = \frac{F}{2\omega_0} \cdot t \cdot \sin(\omega_0 t)$ ⟵ blows up as $t \to \infty$

With damping: $\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = Fe^{i\omega t}$ (or $\cos\omega_0 t$)

$(\omega = \omega_0)$ $\quad x_p = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2}} \cdot \cos(\omega t - \alpha) = \frac{F}{2\omega_0\eta} \cdot \cos(\omega_0 t - \alpha)$

So amplitude $\frac{F}{2\omega_0\eta}$. If damping is small, this blows up, but not all the way to infinity

Does this maximize amplitude?

No. We want $\frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2}}$ to be large, so equivalently

maximize $g_1(\omega) = \frac{1}{(\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2}$ or equivalently,

minimize $g_2(\omega) = (\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2 = (\omega_0^2 - \omega^2) + 4\omega^2\eta^2$

$g_2'(\omega) = 2(\omega_0^2 - \omega^2) \cdot -2\omega + 8\omega\eta^2$

$\Rightarrow -\omega(\omega_0^2 + \omega^2) + 2\omega\eta^2 = 0$

$-\omega_0^2 + \omega^2 + 2\eta^2 = 0$

$\omega^2 = \omega_0^2 - 2\eta^2$

$\omega = \sqrt{\omega_0^2 - 2\eta^2}, \quad \omega_0^2 > 2\eta^2.$

Thus, if $\omega = \sqrt{\omega_0^2 - 2\eta^2}$, $A_{max} = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2}}$

Thus, if $\omega = \sqrt{\omega_0^2 - 2\eta^2}$, $A_{max} = \dfrac{F}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\omega\eta)^2}}$

$$= \frac{F}{\sqrt{\left[\omega_0^2 - (\omega_0^2 - 2\eta^2)\right]^2 + 4\eta^2(\omega_0^2 - 2\eta^2)}}$$

$$= \frac{F}{\sqrt{(2\eta^2)^2 + 4\eta^2(\omega_0^2 - 2\eta^2)}}$$

$$= \frac{F}{\sqrt{4\eta^4 + 4\eta^2\omega_0^2 - 8\eta^4}}$$

$$= \frac{F}{\sqrt{4\eta^2\omega_0^2 - 4\eta^4}} = \frac{F}{2\eta\sqrt{\omega_0^2 - \eta^2}}$$

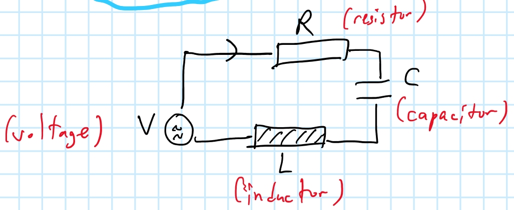
We then say that the forcing function $e^{i\omega t}$ is in resonance with the system.

It doesn't blow up to infinity, but still gets very large.

Note that if η is small, $A_{max} \approx \dfrac{F}{2\eta\omega_0}$, which makes sense

because when η is small, the optimal $\omega = \sqrt{\omega_0^2 - 2\eta^2} \approx \omega_0$.

Ex.



Suppose we have a simple circuit with a fixed voltage source $V = 9$ volts

a resistor with $R = 3$ ohms, inductor $L = 1$ henry, capacitor $C = 0.5$ farads.

What is the current in amps moving across the system at time t.

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}$$

$$\ddot{I} + 3\dot{I} + 2I = 0$$

Char. eq. $m^2 + 3m + 2 = 0$

$m = -1, -2,$

Sol. e^{-t}, e^{-2t}

$\Rightarrow \quad I = c_1 e^{-t} + c_2 e^{-2t} \longrightarrow 0$ as $t \to \infty$

What if the voltage is varying with $V(t) = \sin(t)$?

What if the voltage is varying with $V(t) = \sin(t)$?

$$\ddot{I} + 3\dot{I} + 2I = \cos t = \mathrm{Re}\left(e^{it}\right)$$

homog. soln. $I_c = c_1 e^{-t} + c_2 e^{-2t}$

Use method of undetermined coef. to find particular soln.

$$I_p = ke^{it}, \quad \dot{I}_p = ike^{it}, \quad \ddot{I}_p = -ke^{it}$$

$$-ke^{it} + 3ike^{it} + 2ke^{it} = e^{it}$$

$$k(-1 + 3i + 2) = 1$$

$$k = \frac{1}{1+3i} = \frac{1-3i}{10}$$

$$I_p = \frac{1-3i}{10} e^{it} = \frac{1-3i}{10}\left(\cos t + i\sin t\right)$$

$$\mathrm{Re}(I_p) = \frac{1}{10}\cos t + \frac{3}{10}\sin t$$

What is the maximum amplitude? $\omega_0 = \sqrt{2}$, $\omega = 1$

$$A = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\eta\omega)^2}} = \frac{1}{\sqrt{1 + 9}} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}}$$

Or $\quad A = \left|\frac{1-3i}{10}\right| = \frac{1}{10}\left|1 - 3i\right| = \frac{1}{10}\sqrt{10} = \frac{1}{\sqrt{10}}$

If we want to blow up the system, what forcing function V should we choose?

From above, we know that $\omega = \sqrt{\omega_0^2 - 2\eta^2}$ maximizes amplitude

Let's write our ODE in appropriate form:

$$\ddot{I} + 3\dot{I} + 2I = e^{it}$$

$$\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = f(t)$$

So $\eta = \frac{3}{2}$ and $\omega_0 = \sqrt{2}$

$$\omega = \sqrt{2 - \frac{9}{2}} = \sqrt{-\frac{5}{2}}.$$

But ω is imaginary, so there is no real frequency we could choose to blow up the system.

The resistance in the system is high enough that no frequency gets magnified.